where c_1 is chosen arbitrarily and the remaining coefficients obtained by comparison in the expansion of the two members of the relation[1]

$$g(ax - a^2 + a) = e^{g(x)}.$$

The series will necessarily converge in accordance with the general theory. Since there are two such values, a, there exist two one-parameter families of analytic functions satisfying the problem.

The above solutions are not real functions, so that the existence of a real analytic solution, other than plus infinity, remains to be examined. A smooth curve can be drawn containing one parameter of translation satisfying the problem, except for its possible non-analytic character. Owing to the highly singular character of the point at infinity, in connection with this problem the usual methods cannot be applied to this point. The following method of approach may prove suggestive. Modify the problem so that the fixed point at infinity shall appear at the origin; thus replace f and z by their reciprocals h and t, giving $h(t/(1+t)) = e^{-1/h(t)}$. Choose the parameter of translation so that $h(1) = 1$. Then $h(1/2) = 1/e$, $h(1/3) = 1/e^e \cdots$. At the points where $h(t)$ is thus determined, it approaches zero with extraordinary rapidity as t moves in toward the origin. Use Newton's or some other interpolation formula to obtain a real function $H_1(t)$ coinciding with $h(t)$ at $t = 1, 1/2, 1/3, \cdots, 1/n, \cdots$ and smooth, in the intermediate intervals. Now H_1 may not satisfy the functional relations but $H_1(t)$ and $-1/\log H_1(t/(1+t))$ both coincide with $h(t)$ at the points $t = 1, 1/2, 1/3, \cdots$. The real solution desired may be expected to lie in most places between them, since if they coincide they will form a solution. Take a mean of $H_1(t)$ and $-1/\log H_1(t/(1+t))$ and call it $H_2(t)$. The particular method of choosing a mean is not significant since only a process of successive approximations is attempted. From $H_2(t)$, form $-1/\log H_2(t/(1+t))$; take $H_3(t)$ as a mean of these and proceed thus indefinitely. Inspection would suggest that the process might be arranged to lead to a determinate real analytic function in the limit, which function $H(t)$ would coincide with $-1/\log H(t/(1+t))$, so that $H(t)$ is a solution, $h(t)$, desired for the modified problem. Thence, the solution, $f(z)$, of the given problem is obtained by taking reciprocals. The above method lends itself to numerical handling, but the proof of the convergence to an analytic function will, of course, involve the usual theoretical complications.

2862 [1920, 428]. Proposed by J. L. RILEY, Stephensville, Texas.

Show that the whole area commanded by a gun on a hillside is an ellipse whose focus is at the gun, whose eccentricity is the sine of the inclination of the hill to the horizon, and whose semi-latus rectum is twice the greatest height to which the gun could send a ball.

SOLUTION BY A. V. RICHARDSON, Bishop's College, Lennoxville, Quebec.

Let G be the position of the gun, AGA' the line of greatest slope, α the inclination of the hillside, and GL the maximum range in a direction making an angle θ with the line of greatest slope.

Also let β be the angle which GL makes with its projection GN on the horizontal plane through G. Then if u, $\phi + \beta$ represent the muzzle velocity and angle of elevation, respectively, for the maximum range GL we have

$$\begin{aligned} u\cos(\phi+\beta)\cdot t &= GN \qquad (t = \text{time of flight}) \\ &= GL\cos\beta. \qquad (1) \end{aligned}$$

[1] We may use the equation $ag'(ax - a^2 + a) = g'(x)g(ax - a^2 + a)$, g' denoting derivative. We shall find each c equal to a polynomial in the c's with lower suffix; in fact, c_n will be equal to c_1^n times a function of a alone, so that we can put $c_1 = 1$ in these equations. We shall have $2(a-1)c_2 = 1$, and in general

$$(n+1)(a^n - 1)c_{n+1} = a^{n-1}c_n + 2a^{n-2}c_2c_{n-1} + \cdots + (n-1)ac_{n-1}c_2 + nc_n.$$

Now suppose $|c_r| \leqq k^{r-1}$, $r = 2, 3, \cdots n$. Then

$$(n+1)|a^n - 1|\,|c_{n+1}| \leqq [|a|^{n-1} + 2|a|^{n-2} + \cdots + n]k^{n-1} < (n+1)\frac{|a|^n - 1}{|a| - 1}k^{n-1}.$$

Also $|a^n - 1| \geqq |a|^n - 1$, and hence $|c_{n+1}| \leqq k^{n-1}/(|a| - 1)$.

Furthermore, $|c_2| < 1/(|a| - 1)$. Therefore, if we take $k \geqq 1/(|a| - 1)$, we shall have for all values of n $|c_n| \leqq k^{n-1}$.

The radius of convergence of the series for $g(x)$ is at least equal to $1/k|c_1|$ or $(|a| - 1)/|c_1|$. —EDITOR.

Also resolving perpendicular to GL

$$0 = u \sin \phi \cdot t - \tfrac{1}{2} g \cos \beta \cdot t^2. \tag{2}$$

Hence from (1) and (2) $GL = [2u^2 \cos(\phi + \beta) \sin \phi]/[g \cos^2 \beta]$ when the expression on the right-hand side is a maximum, *i.e.*, $GL = u^2[\sin(2\phi + \beta) - \sin \beta]/[g \cos^2 \beta]$; whence $2\phi + \beta = \pi/2$, and

$$GL = \frac{u^2(1 - \sin \beta)}{g \cos^2 \beta} = \frac{u^2}{g(1 + \sin \beta)}. \tag{3}$$

Again, if LE is drawn perpendicular to GA, and GF is the horizontal projection of GE, we have

$$\sin \beta = \frac{NL}{GL} = \frac{NL}{GE} \cdot \frac{GE}{GL} = \frac{FE}{GE} \cdot \frac{GE}{GL} = \sin \alpha \cdot \cos \theta.$$

Hence, (3) becomes $(u^2/g)/GL = 1 + \sin \alpha \cos \theta$, *i.e.*, L is on an ellipse, focus G, eccentricity $\sin \alpha$, and the semi-latus rectum u^2/g.

REMARK BY OTTO DUNKEL, Washington University—This problem may also be solved by finding the intersection of the envelope of the trajectories (see Granville, *Calculus*, first edition, p. 216) $z = (u^2/2g) - (g/2u^2)(x^2 + y^2)$ and the inclined plane $z = y \tan \alpha$.

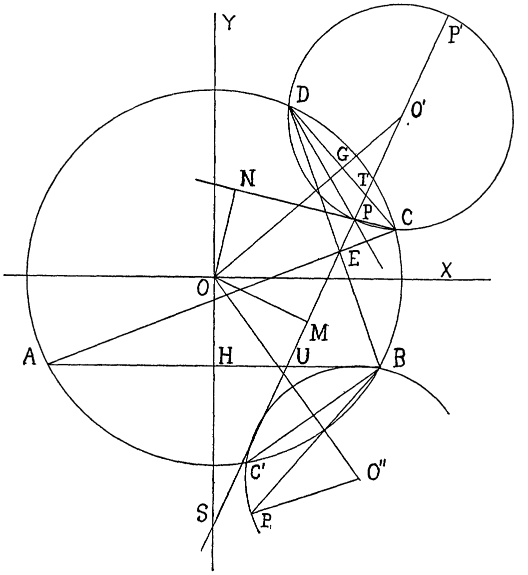
Also solved by AUGUSTUS BOGARD, A. M. HARDING, and WILLIAM HOOVER.

2865 [1920, 482]. Proposed by JOSEPH ROSENBAUM, Milford, Conn.

In a circle, a chord AB is fixed in position and a moving chord CD is constant in length. Find the locus of the intersection of the bisectors of the angles ACD and BDC.

I. SOLUTION BY F. L. WILMER, Omaha, Neb.

A solution of the problem in all its ramifications requires a discussion of a number of distinct cases, segregable into two classes according as the lines connecting the extremities of the two chords do or do not cross within the given circle. One special case of the first class has been selected here for discussion to show a workable method of attack of the problem in the various cases.



Let the radius of the given circle be the unit of length and suppose that the internal bisectors of the angles of the triangle CDE (see figure) meet in P, a point of the locus. Let the center O of the unit circle be the origin of rectangular coördinates with the x-axis parallel to AB. It will be seen at once that the acute angle b which PC makes with DP is constant, so that P, C, D, and P' (the intersection of the two external bisectors of the angles C and D) lie upon a circle of constant radius with center at O'. The central angle $DO'C = 2b$. Denote the lengths of the arcs AB and CD of the fixed circle by c and a, the angle YOO' by φ. When $\varphi = 0$, P lies on the y-axis and the angle $OO'P$ is also zero. When CD takes the position $C'D'$, $D' = B$, $\angle\, OO'P$ becomes in the new position at O'' $\angle\, OO''P_1 = \angle OO''C' + \angle\, C'O''P_1$ $= b + 2\angle\, C'D'P_1 = b + a/2$. The angle φ is now seen from the figure to be $\angle\, YOO'' = \angle\, YOB + a/2 = \pi - c/2 + a/2 = 2b + a$, since $a + c + 4b = 2\pi$. Hence $\angle\, YOO'' = 2\,\angle\, OO''P_1$ and it may be shown that for any position of P this relation is true. Hence $\angle\, OO'P = \varphi/2$.